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# STABILITY OF RANK 2 ULRICH BUNDLES ON PROJECTIVE $K3$ SURFACE

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ABSTRACT. Let  $F \subseteq \mathbb{P}^{a+1}$  be a non-degenerate  $K3$  surface of degree  $2a$ , where  $a \geq 2$ . In this paper we deal with Ulrich bundles on  $F$  of rank 2. We deal with their stability and we construct  $K3$  surfaces endowed with families of non-special Ulrich bundles of rank 2 for each  $a \geq 2$ .

## 1. INTRODUCTION AND NOTATION

Throughout the whole paper,  $\mathbb{P}^N$  will denote the projective space of dimension  $N$  over the complex field  $\mathbb{C}$ .

Each smooth surface  $F \subseteq \mathbb{P}^N$  is endowed with a polarization  $\mathcal{O}_F(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_F$ . A natural problem in the study of the geometry of  $F$  is to deal with the vector bundles that it supports.

Clearly we can restrict our attention to *indecomposable* bundles, i.e. bundles which do not split as a direct sum of bundles of lower rank. From the cohomological viewpoint, the simplest vector bundles are the *arithmetically Cohen–Macaulay* (aCM for short) ones, i.e. bundles  $\mathcal{E}$  such that  $h^1(F, \mathcal{E}(th)) = 0$  for  $t \in \mathbb{Z}$ . Notice that such a property is trivially invariant up to shifting degrees. Thus we can focus on *initialized* bundles, i.e. bundles  $\mathcal{E}$  such that  $h^0(F, \mathcal{E}(-h)) = 0$  and  $h^0(F, \mathcal{E}) \neq 0$ .

Horrocks theorem (see [21] and the references therein) guarantees that  $\mathcal{O}_F$  is the unique initialized, indecomposable, aCM bundle when  $F \subseteq \mathbb{P}^N$  is a plane. Recall that a closed subscheme  $F \subseteq \mathbb{P}^N$  is aCM if it is projectively normal and  $\mathcal{O}_F$  is aCM. A very general result of D. Eisenbud and D. Herzog (see [12]) implies that, besides planes, only few other surfaces support at most a finite number of aCM bundles, namely smooth quadrics, smooth rational scrolls of degree up to 4, the Veronese surface.

M. Casanellas and R. Hartshorne proved in [3] and [4] that a smooth cubic surface in  $\mathbb{P}^3$  is endowed with families of arbitrary dimension of non-isomorphic, indecomposable initialized aCM bundles. In order to achieve their results, the authors constructed families of initialized aCM bundles with an extra property. Indeed they are *Ulrich bundles*, i.e. bundles  $\mathcal{E}$  on  $F$  whose minimal free resolution as sheaves on  $\mathbb{P}^3$  is linear. To complete the picture in the case of a cubic surface, we recall that D. Faenzi completely described aCM bundles of ranks 1 and 2 in [14].

Some results are known also for quartic surfaces  $F \subseteq \mathbb{P}^3$ . K. Watanabe classified in [24] aCM line bundles on  $F$ , identifying Ulrich line bundles. E. Coskun, R. Kulkarni, Y. Mustopa proved in [10] that such an  $F$  always supports a family of dimension 14 of Ulrich bundles of rank 2 with first Chern class  $\mathcal{O}_F(3h)$ . As a consequence of the existence of these

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bundles one can also infer that  $F$  is linear pfaffian, i.e. the quartic polynomial defining  $F$  is the pfaffian of a  $8 \times 8$  skew-symmetric matrix with linear entries.

Moreover, another almost immediate application of this existence result is that such a surface also supports families of arbitrary dimension of non-isomorphic, indecomposable Ulrich bundles (see the note [7]). In [8] other interesting families of initialized aCM bundles of rank 2 are constructed: their existence implies that  $F$  is quadratic pfaffian, i.e. the quartic polynomial defining  $F$  is also the pfaffian of a  $4 \times 4$  skew-symmetric matrix with quadratic entries. Finally, the complete description of initialized aCM bundles of rank 2 on each linear determinantal smooth quartic surface, i.e. a surface defined by a quartic polynomial which is the determinant of a  $4 \times 4$  matrix with linear entries, is exploited in [6]. As far as we know there are no other general results for smooth quartic surfaces.

When the codimension increases the picture becomes quickly vague. E.g. even for del Pezzo surfaces only scattered results are known: for this class of surfaces J. Pons Lopes and F. Tonini studied aCM line bundles in [22], while in [11] the authors gave, among the other results, restrictions on the first Chern class of Ulrich bundles.

Notice that quartic surfaces are a particular case of *K3 surfaces*, i.e. smooth regular surfaces  $F$  such that  $\omega_F \cong \mathcal{O}_F$ . These surfaces can be embedded in  $\mathbb{P}^{a+1}$  as a non-degenerate aCM surface of degree  $2a$  for some  $a \geq 2$  (see [23] for the details). M. Aprodu, G. Farkas, A. Ortega generalized in [1] the results of [10] to this family of *K3 surfaces*  $F$ , under an extra technical condition. They construct therein families of rank 2 Ulrich bundles with first Chern class  $\mathcal{O}_F(3h)$ : following Proposition 6.2 of [13], they call such bundles *special*.

As we already pointed out, the role of Ulrich bundles is particularly important, hence we ask for further informations on them. For example, it would be interesting to answer the following questions.

- (1) Are there restrictions on the Chern classes of Ulrich bundles on a *K3 surface*?
- (2) Are there other rank 2 Ulrich bundles on a *K3 surface* besides the special ones described in [10] and [1]?
- (3) Which are the (semi)stability properties of a rank 2 Ulrich bundle?

In this paper we give partial answers to the questions listed above. E.g., at the end of Section 4 we prove the following easy proposition answering the first question (see Proposition 4.2).

**Partial answer to question 1.** *Let  $\mathcal{E}$  be a Ulrich bundle of rank  $r$  on a non-degenerate *K3 surface*  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$ , where  $a \geq 2$ .*

*Then  $c_1(\mathcal{E})^2$  is an even integer satisfying*

$$4(a-1)r^2 \leq c_1(\mathcal{E})^2 \leq \frac{9}{2}ar^2,$$

*where  $c_1(\mathcal{E})^2 \neq \frac{9}{2}ar^2 - 2$  if  $r$  is even. Moreover,  $c_1(\mathcal{E})^2 = \frac{9}{2}ar^2$  if and only if  $c_1(\mathcal{E}) = 3rh/2$ .*

*If  $\mathcal{E}$  is simple, then  $(4a-2)r^2 - 2 \leq c_1(\mathcal{E})^2$ .*

We also show that both the above upper and lower bounds are trivially sharp. Moreover, in Section 5, we are also able to answer the second question, proving that, when we restrict to bundles of rank 2, all the intermediate values of the above inequality are actually attained (see Theorem 5.3) for suitable *K3 surfaces*. The bundles that we construct are non-special in the sense of [1], i.e. their first Chern class is not  $\mathcal{O}_F(3h)$ .

**Partial answer to question 2.** *Let  $a \geq 2$ . For each choice of an integer  $u$  in the range  $4a-1 \leq u \leq 5a+4$ ,  $u \neq 5a+3$ , there exists a non-degenerate *K3 surface*  $F \subseteq \mathbb{P}^{a+1}$*

of degree  $2a$  and  $\mathrm{rk}(\mathrm{Pic}(F)) = 3$  supporting a simple Ulrich bundle  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  and  $c_2(\mathcal{E}) = u$ .

Notice that when  $\mathrm{rk}(\mathrm{Pic}(F)) = 1$  only the bundles with  $u = 5a + 4$  can actually exist on  $F$  and they are exactly the aforementioned special bundles constructed in [10] and [1] (see [9]: see also [5]).

Our construction cannot be extended for obtaining bundles on a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  with  $\mathrm{rk}(\mathrm{Pic}(F)) = 2$ : in particular we are unfortunately unable to prove or disprove the existence of these bundles on surfaces satisfying  $\mathrm{rk}(\mathrm{Pic}(F)) = 2$ . Moreover, as pointed out in Section 2 of [13], the Chow forms of the surfaces we use are always linear determinantal. Thus the problem of the existence of non-special rank 2 Ulrich bundles on  $K3$  surfaces whose Chow form is not determinantal remains wide open.

Finally, in Section 6, we answer the third question raised above. Indeed we prove therein the following result (see Theorem 6.2).

**Partial answer to question 3.** *Let  $F \subseteq \mathbb{P}^{a+1}$  be a non-degenerate  $K3$  surface of degree  $2a$ , where  $a \geq 2$ .*

*If  $\mathcal{E}$  is an indecomposable Ulrich bundle of rank 2 on  $F$  which is strictly semistable and general in its moduli space, then it fits into an exact sequence of the form*

$$(1) \quad 0 \longrightarrow \mathcal{O}_F(A) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(B) \longrightarrow 0.$$

*where  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are Ulrich line bundles on  $F$  such that  $AB = 4a - 1$ . In particular  $c_1(\mathcal{E})^2 = 16a - 10$  and  $c_2(\mathcal{E}) = 4a - 1$ .*

It follows that the general Ulrich bundle  $\mathcal{E}$  of rank 2 with fixed Chern classes  $c_1(\mathcal{E})$  and  $c_2(\mathcal{E})$  on a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  is always stable when  $c_2(\mathcal{E}) \neq 4a - 1$ . Also in this case our answer is partial: indeed we are not able to deal with the stability properties of each general Ulrich bundle  $\mathcal{E}$  such that  $c_2(\mathcal{E}) = 4a - 1$ .

In Section 2 we recall the results that we need in the paper on Ulrich bundles. In Section 3 we summarize several facts about  $K3$  surfaces. Section 4 is devoted to list and inspect some properties of Ulrich bundles on  $K3$  surfaces. In Section 5 we focus on rank 2 bundles. Finally, in Section 6 we deal with the stability properties of rank 2 Ulrich bundles.

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For all the notations and unproven results we refer to [15].

## 2. GENERAL RESULTS ON ULRICH BUNDLES

In this section we summarize some general results on Ulrich bundles on a smooth, irreducible, closed subscheme  $X \subseteq \mathbb{P}^N$ . In what follows we will always set  $\mathcal{O}_X(h) := \mathcal{O}_{\mathbb{P}^N}(1) \otimes \mathcal{O}_X$ .

**Definition 2.1.** Let  $X \subseteq \mathbb{P}^N$  be a smooth irreducible closed subscheme and let  $\mathcal{F}$  be a vector bundle on  $X$ .

We say that:

- $\mathcal{F}$  is initialized if  $h^0(X, \mathcal{F}(-h)) = 0 \neq h^0(X, \mathcal{F})$ .
- $\mathcal{F}$  is aCM if  $h^i(X, \mathcal{F}(th)) = 0$  for each  $t \in \mathbb{Z}$  and  $i = 1, \dots, \dim(X) - 1$ .
- $\mathcal{F}$  is Ulrich if  $h^i(X, \mathcal{F}(-ih)) = h^j(X, \mathcal{F}(-(j+1)h)) = 0$  for each  $i > 0$  and  $j < \dim(X)$ .

Ulrich bundles collect many interesting properties (see Section 2 of [13]). E.g. they are automatically initialized, aCM and globally generated. Every direct summand of a Ulrich bundle is Ulrich as well. Finally, as already pointed out in the introduction,  $\mathcal{F}$  is Ulrich if

and only if it has a linear minimal free resolution over  $\mathbb{P}^N$ . Ulrich bundles also well behave with respect to the notions of (semi)stability and  $\mu$ -(semi)stability. Recall that for each bundle  $\mathcal{F}$  on  $X$ , the slope  $\mu(\mathcal{F})$  and the reduced Hilbert polynomial  $p_{\mathcal{F}}(t)$  (with respect to  $\mathcal{O}_X(h)$ ) are defined as follows:

$$\mu(\mathcal{F}) = c_1(\mathcal{F})h^{\dim(X)-1}/\mathrm{rk}(\mathcal{F}), \quad p_{\mathcal{F}}(t) = \chi(\mathcal{F}(th))/\mathrm{rk}(\mathcal{F}).$$

The bundle  $\mathcal{F}$  is  $\mu$ -semistable (resp.  $\mu$ -stable) if for all subsheaves  $\mathcal{G}$  with  $0 < \mathrm{rk}(\mathcal{G}) < \mathrm{rk}(\mathcal{F})$  we have  $\mu(\mathcal{G}) \leq \mu(\mathcal{F})$  (resp.  $\mu(\mathcal{G}) < \mu(\mathcal{F})$ ).

The bundle  $\mathcal{E}$  is called semistable (resp. stable) if for all  $\mathcal{G}$  as above  $p_{\mathcal{G}}(t) \leq p_{\mathcal{E}}(t)$  (resp.  $p_{\mathcal{G}}(t) < p_{\mathcal{E}}(t)$ ) for  $t \gg 0$ . We recall that in order to check the semistability and stability of a bundle one can restrict the attention only to subsheaves such that the quotient is torsion-free.

The following chain of implications holds for  $\mathcal{F}$ :

$$\mathcal{F} \text{ is } \mu\text{-stable} \Rightarrow \mathcal{F} \text{ is stable} \Rightarrow \mathcal{F} \text{ is semistable} \Rightarrow \mathcal{F} \text{ is } \mu\text{-semistable}.$$

For the following result see Theorem 2.9 of [4].

**Theorem 2.2.** *Let  $X \subseteq \mathbb{P}^N$  be a smooth, irreducible closed subscheme.*

*If  $\mathcal{E}$  is a Ulrich bundle on  $X$  the following assertions hold.*

- (1)  $\mathcal{E}$  is semistable and  $\mu$ -semistable.
- (2)  $\mathcal{E}$  is stable if and only if it is  $\mu$ -stable.
- (3) If

$$(2) \quad 0 \longrightarrow \mathcal{L} \longrightarrow \mathcal{E} \longrightarrow \mathcal{M} \longrightarrow 0$$

*is an exact sequence of coherent sheaves with  $\mathcal{M}$  torsion free and  $\mu(\mathcal{L}) = \mu(\mathcal{E})$ , then both  $\mathcal{L}$  and  $\mathcal{M}$  are Ulrich bundles.*

We conclude this section with the following helpful result.

**Lemma 2.3.** *Let  $X \subseteq \mathbb{P}^N$  be a smooth, irreducible closed subscheme with  $h^1(X, \mathcal{O}_X) = 0$ .*

*If  $\mathcal{E}$  is a Ulrich bundle of rank 2 on  $X$ , then  $\mathcal{E}$  is simple if and only if it is indecomposable.*

*Proof.* If  $\mathcal{E}$  is simple, then it is trivially indecomposable. Conversely, assume that  $\mathcal{E}$  is indecomposable. If it is  $\mu$ -stable, then it is simple (see [17], Corollary 1.2.8).

Assume that  $\mathcal{E}$  is strictly semistable. In particular  $\mathcal{E}$  fits into Sequence (2) with  $\mathcal{M}$  torsion-free and  $\mu(\mathcal{L}) = \mu(\mathcal{E})$ . It follows from Theorem 2.2 that  $\mathcal{L} \cong \mathcal{O}_X(A)$  and  $\mathcal{M} \cong \mathcal{O}_X(B)$  are Ulrich line bundles,  $\mu(\mathcal{O}_X(A)) = \mu(\mathcal{O}_X(B))$ . Thus  $\mathcal{E}$  fits into the exact sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(B) \longrightarrow 0.$$

If  $\mathcal{O}_X(A) \cong \mathcal{O}_X(B)$  the above sequence would split because  $h^1(X, \mathcal{O}_X(A-B)) = h^1(X, \mathcal{O}_X) = 0$  in this case. Thus we can assume  $\mathcal{O}_X(A) \not\cong \mathcal{O}_X(B)$ .

The above sequence splits if and only if it corresponds to  $0 \in \mathrm{Ext}_F^1(\mathcal{O}_X(B), \mathcal{O}_X(A))$ , thus if and only if  $\mathcal{E}$  is not simple, due to Proposition 5.3 of [22]. Since we are assuming that  $\mathcal{E}$  is indecomposable, it follows that  $\mathcal{E}$  is simple.  $\square$

### 3. GENERAL RESULTS ON $K3$ SURFACES

We recall some facts on a  $K3$  surface  $F \subseteq \mathbb{P}^N$  with hyperplane line bundle  $\mathcal{O}_F(h)$ . They are collected from several places (e.g. see [23] and [16]).

We know that  $\omega_F \cong \mathcal{O}_F$  and  $q(F) = 0$ . In particular  $p_a(F) = p_g(F) = 1$ . The first important fact is that Serre duality for each locally free sheaf  $\mathcal{F}$  on  $F$  becomes

$$h^i(F, \mathcal{F}) = h^{2-i}(F, \mathcal{F}^\vee), \quad i = 0, 1, 2.$$

Moreover Riemann–Roch theorem on  $F$  is

$$(3) \quad h^0(F, \mathcal{F}) + h^2(F, \mathcal{F}) = h^1(F, \mathcal{F}) + 2\mathrm{rk}(\mathcal{F}) + \frac{c_1(\mathcal{F})^2}{2} - c_2(\mathcal{F}).$$

In particular, if  $\mathcal{F} \cong \mathcal{O}_F(D)$  for a divisor  $D$  with  $D^2 \geq -2$ , then either  $D$  or  $-D$  is necessarily effective.

If  $D$  is an effective divisor on  $F$ , then  $h^2(F, \mathcal{O}_F(D)) = h^0(F, \mathcal{O}_F(-D)) = 0$ . Moreover

$$h^1(F, \mathcal{O}_F(D)) = h^1(F, \mathcal{O}_F(-D)) = h^0(D, \mathcal{O}_D) - 1$$

(see [23], Lemma 2.2). It follows that

$$(4) \quad h^0(F, \mathcal{O}_F(D)) = 2 + \frac{D^2}{2}, \quad \deg(D) = Dh, \quad p_a(D) = 1 + \frac{D^2}{2},$$

for each integral divisor  $D$  on  $F$  (see [23], Paragraph 2.4). In particular, the integral fixed divisors  $D$  satisfy  $D^2 = -2$  and  $D \cong \mathbb{P}^1$ .

We summarize the other helpful results we will need in the following statement.

**Proposition 3.1.** *Let  $F$  be a  $K3$  surface.*

*For each effective divisor  $D$  on  $F$  such that  $|D|$  has no fixed components the following assertions hold.*

- (1)  $D^2 \geq 0$  and  $\mathcal{O}_F(D)$  is globally generated.
- (2) If  $D^2 > 0$ , then the general element of  $|D|$  is irreducible and smooth: in this case  $h^1(F, \mathcal{O}_F(D)) = 0$ .
- (3) If  $D^2 = 0$ , then there is an irreducible divisor  $\overline{D}$  with  $p_a(\overline{D}) = 1$  such that  $\mathcal{O}_F(D) \cong \mathcal{O}_F(e\overline{D})$  where  $e - 1 := h^1(F, \mathcal{O}_F(D))$ : in this case the general element of  $|D|$  is smooth.

*Proof.* See [23], Proposition 2.6 and Corollary 3.2. □

Now let  $h^2 = 2a$  with  $a \geq 2$ . The induced embedding  $F \subseteq \mathbb{P}^{a+1}$  is non-degenerate and projectively normal thanks to [23], Theorem 6.1. Moreover,  $\mathcal{O}_F(th)$  is very ample too for each  $t \geq 1$ , hence by Serre duality we have  $h^1(F, \mathcal{O}_F(-th)) = h^1(F, \mathcal{O}_F(th)) = 0$  in the same range. Since we have  $h^1(F, \mathcal{O}_F) = 0$  by definition, it follows that the embedding  $F \subseteq \mathbb{P}^{a+1}$  is actually aCM.

It would be interesting to classify aCM line bundles on the embedded surface. When  $a = 2$  a complete classification can be found in [24]. A similar classification for double covers of  $\mathbb{P}^2$  can be found in [25]. The problem of identifying aCM line bundles on  $K3$  surfaces is by no way trivial, as one can check by looking at the quoted papers.

#### 4. ULRICH BUNDLES ON $K3$ SURFACES

In this section we will prove some general preliminary results about Ulrich bundles on non-degenerate  $K3$  surfaces  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$ , where  $a \geq 2$ , giving a partial answer to the first question raised in the introduction.

**Lemma 4.1.** *Let  $F \subseteq \mathbb{P}^{a+1}$  be a non-degenerate  $K3$  surface of degree  $2a$ , where  $a \geq 2$ .*

*The following assertions are equivalent for a vector bundle  $\mathcal{E}$  of rank  $r$  on  $F$ :*

- (1)  $\mathcal{E}$  is Ulrich;

- (2)  $\mathcal{E}^\vee(3h)$  is Ulrich;  
 (3)  $\mathcal{E}$  is aCM and

$$(5) \quad c_1(\mathcal{E})h = 3ar, \quad c_2(\mathcal{E}) = \frac{c_1(\mathcal{E})^2}{2} - 2(a-1)r;$$

- (4)  $h^0(F, \mathcal{E}(-h)) = h^0(F, \mathcal{E}^\vee(2h)) = 0$  and Equalities (5) hold.

*Proof.* If  $\mathcal{E}$  is Ulrich then the same is true for  $\mathcal{E}^\vee(3h)$  due to [11], Proposition 2.11. Since  $(\mathcal{E}^\vee(3h))^\vee(3h) \cong \mathcal{E}$ , it follows that the converse is true as well. Thus the first and second assertion are equivalent.

The equivalence of the third and first assertions is [11], Propositions 2.10.

Assume that  $\mathcal{E}$  is aCM and Equalities (5) hold. Thus  $h^1(F, \mathcal{E}(-th)) = 0$  for  $t = 1, 2$ , hence Formula (3) and Equalities (5) imply

$$\begin{aligned} h^0(F, \mathcal{E}(-h)) &\leq \chi(\mathcal{E}(-h)) = 0, \\ h^0(F, \mathcal{E}^\vee(2h)) &= h^2(F, \mathcal{E}(-2h)) \leq \chi(\mathcal{E}(-2h)) = 0. \end{aligned}$$

Finally, assume that  $h^0(F, \mathcal{E}(-h)) = h^0(F, \mathcal{E}^\vee(2h)) = 0$ , whence  $h^2(F, \mathcal{E}(-2h)) = h^0(F, \mathcal{E}^\vee(2h)) = 0$ . We have

$$\begin{aligned} h^0(F, \mathcal{E}(-2h)) &\leq h^0(F, \mathcal{E}(-h)) \\ h^2(F, \mathcal{E}(-h)) &= h^0(F, \mathcal{E}^\vee(h)) \leq h^0(F, \mathcal{E}^\vee(2h)) = h^2(F, \mathcal{E}(-2h)). \end{aligned}$$

It follows that  $h^1(F, \mathcal{E}(-th)) = -\chi(\mathcal{E}(-th))$ , for  $t = 1, 2$ . Formula (3) and Equalities (5) yield  $h^1(F, \mathcal{E}(-th)) = 0$ . We deduce that  $\mathcal{E}$  is Ulrich.  $\square$

Notice that if  $r = 1$ , then  $\mathcal{O}_F(D)$  is Ulrich if and only if  $D^2 = 4(a-1)$ ,  $Dh = 3a$  and  $h^0(F, \mathcal{O}_F(D-h)) = h^0(F, \mathcal{O}_F(2h-D)) = 0$ .

Let  $\mathcal{E}$  be a Ulrich bundle of rank  $r$  on  $F$  with Chern classes  $c_1$  and  $c_2$ . Thanks to Theorem 2.2,  $\mathcal{E}$  is  $\mu$ -semistable, thus Bogomolov inequality (see [17], Theorem 3.4.1) holds for  $\mathcal{E}$ . Taking into account of Equalities (5) we thus obtain  $c_1^2 - 4(a-1)r^2 \geq 0$ .

Since  $\mathcal{E}$  is globally generated, it follows that the same is true for  $\det(\mathcal{E})$ . We know that  $\det(\mathcal{E}) \neq \mathcal{O}_F$ , because  $\mu(\mathcal{E}) = 3ar$ . Thus there is a smooth curve  $C \subseteq F$  such that  $\det(\mathcal{E}) \cong \mathcal{O}_F(C)$ . Trivially we have  $Ch = 3ar$  and  $C^2 = c_1^2$ . Hodge index theorem applied to  $h$  and  $C$  yields

$$2ac_1^2 - 9a^2r^2 = \begin{vmatrix} 2a & 3ar \\ 3ar & C^2 \end{vmatrix} \leq 0,$$

hence  $c_1^2 \leq \frac{9}{2}ar^2$ . Moreover, equality holds if and only if  $c_1 = 3rh/2$ , because  $\text{Num}(F) \cong \text{Pic}(F)$  on a  $K3$  surface (see [16], Proposition 1.2.5) and  $\det(\mathcal{E})$  is globally generated. Finally recall that  $c_1^2$  is necessarily even.

Assume now that  $\mathcal{E}$  is also simple, thus the coarse moduli space  $\text{Spl}_F(r; c_1, c_2)$  for rank  $r$  simple, vector bundles on  $F$  with Chern classes  $c_1$  and  $c_2$  has a non-empty component containing  $\mathcal{E}$ . As pointed out in Theorem 0.1 of [20], thanks to Equalities (5), such a component has dimension  $c_1^2 - (4a-2)r^2 + 2 \geq 0$ . Thus if  $\mathcal{E}$  is an indecomposable rank 2 Ulrich bundle on  $F$  the above lower bound for  $c_1^2$  can be strengthened as follows:  $(4a-2)r^2 - 2 \leq c_1^2$ .

We can summarize the above computations in the first pair of inequalities of the following statement.

**Proposition 4.2.** *Let  $\mathcal{E}$  be a Ulrich bundle of rank  $r$  on a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$ , where  $a \geq 2$ .*

*Then  $c_1(\mathcal{E})^2$  is an even integer satisfying*

$$4(a-1)r^2 \leq c_1(\mathcal{E})^2 \leq \frac{9}{2}ar^2,$$

*where  $c_1(\mathcal{E})^2 \neq \frac{9}{2}ar^2 - 2$  if  $r$  is even. Moreover,  $c_1(\mathcal{E})^2 = \frac{9}{2}ar^2$  if and only if  $c_1(\mathcal{E}) = 3rh/2$ .*

*If  $\mathcal{E}$  is simple, then  $(4a-2)r^2 - 2 \leq c_1(\mathcal{E})^2$ .*

*Proof.* We have to prove that there are no indecomposable Ulrich bundles  $\mathcal{E}$  of even rank  $r = 2s$  such that  $c_1(\mathcal{E})^2 = \frac{9}{2}ar^2 - 2$ .

Assume that such a bundle  $\mathcal{E}$  actually exists. The line bundle  $\det(\mathcal{E})$  is globally generated, because the same is true for  $\mathcal{E}$ . Let  $C \in |\det(\mathcal{E})|$  be a smooth integral curve. Since  $Ch = 3ar$  and  $C^2 = \frac{9}{2}ar^2 - 2$ , it follows that  $C$  is a curve of degree  $6as$  and genus  $9as^2$ .

Thus  $\omega_C(-3sh)$  cannot be effective, hence Riemann–Roch theorem on  $C$  yields

$$h^0(C, \mathcal{O}_C(3sh)) = 9as^2 + 1.$$

The cohomology of sequence

$$0 \longrightarrow \mathcal{I}_{C|F}(3sh) \longrightarrow \mathcal{O}_F(3sh) \longrightarrow \mathcal{O}_C(3sh) \longrightarrow 0$$

and the first Equality 4 for  $\mathcal{O}_F(3sh)$  imply the existence of a surface  $S$  of degree  $3s$  intersecting  $F$  along a curve containing  $C$ . Since  $\deg(S \cap F) = 6as$  it follows that  $C = S \cap F$ . Adjunction formula on  $F$  now would imply that the genus of  $C$  would be  $9as^2 + 1$ , a contradiction.  $\square$

**Remark 4.3.** The bounds above are sharp in many cases.

Indeed let  $F$  support a Ulrich line bundle  $D$ , so that  $D^2 = 4(a-1)$  and  $Dh = 3a$  (see Equalities (5)). Thus  $\mathcal{E} := \mathcal{O}_F(D)^{\oplus s}$  is a Ulrich bundle of rank  $r := s$  with  $c_1(\mathcal{E})^2 = 4(a-1)r^2$ .

Similarly, every general non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  supports an indecomposable Ulrich bundle  $\mathcal{F}$  of rank 2 with  $c_1(\mathcal{E}) = 3h$  (see [1], Theorem 0.4). It is easy to check that  $\mathcal{E} := \mathcal{F}^{\oplus s}$  is a Ulrich bundle of rank  $r := 2s$  with  $c_1(\mathcal{E})^2 = \frac{9}{2}ar^2$ .

Obviously, such bundles are not simple, unless  $s = 1$  and  $\mathcal{F}$  is indecomposable.

On the one hand, we will see in the next section that the above bound is optimal when  $\mathcal{E}$  is a Ulrich bundle of rank  $r = 2$ . On the other hand when  $\mathcal{E}$  is a Ulrich bundle of odd rank  $r$ , the upper bound  $c_1(\mathcal{E})^2 \leq 9r^2$  of Proposition 4.2 is never sharp, because  $D^2$  is even for each divisor  $D$  on  $F$ . Actually, for  $r = 1$ , we can certainly say that such a bound is very far from optimality. Thus the following question raises naturally.

**Question 4.4.** *When  $r$  is odd, is there any bound which is sharper than the one given in Proposition 4.2?*

## 5. ULRICH BUNDLES OF RANK 2 ON $K3$ SURFACES

The second question raised in the introduction is to prove whether indecomposable Ulrich bundles satisfying the above bounds actually exist. We will give below a partial answer to this question, by constructing explicitly  $K3$  surfaces endowed with suitable Ulrich bundles of rank 2.

Let  $\mathcal{E}$  be a Ulrich bundle on  $F$  of rank 2. We have

$$c_1(\mathcal{E})h = 6a, \quad c_2(\mathcal{E}) = \frac{c_1^2(\mathcal{E})}{2} - 4(a-1)$$



(see Equalities (5)), hence  $\mu(\mathcal{E}) = 3a$ .

It follows from Lemma 2.3 and Proposition 4.2 that an indecomposable rank 2 Ulrich bundle  $\mathcal{E}$  is simple, hence  $c_1(\mathcal{E})^2$  is an even integer satisfying

$$16(a-1) \leq c_1(\mathcal{E})^2 \leq 18a, \quad c_1(\mathcal{E})^2 \neq 18a-2.$$

We already know that both the cases  $c_1(\mathcal{E})^2 = 16(a-1)$  and  $18a$  occur (see Remark 4.3).

Assume the existence of a Ulrich bundle  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 16a-14$  (resp.  $16a-12$ ). Due to Proposition 4.2, the bundle  $\mathcal{E}$  is not simple. Thus Lemma 2.3 implies that  $\mathcal{E} \cong \mathcal{O}_F(A) \oplus \mathcal{O}_F(B)$ , hence  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are both Ulrich. Equalities (5) yield  $A^2 = B^2 = 4(a-1)$ : the equality  $c_1(\mathcal{E})^2 = 16a-14$  (resp.  $16a-12$ ) finally forces  $AB = 4a-3$  (resp.  $AB = 4a-2$ ).

Assume that  $AB = 4a-3$ . Thus  $A(3h-B) = 5a+3$ , hence  $\bar{\mathcal{E}} := \mathcal{O}_F(A) \oplus \mathcal{O}_F(3h-B)$  would be a Ulrich bundle with  $c_1(\bar{\mathcal{E}})^2 = (A+B)^2 = 18a-2$ , contradicting Proposition 4.2. We conclude that such a case cannot occur.

We will now show that also all the other remaining cases occur, in the sense that there is a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  with Picard number  $\text{rk}(\text{Pic}(F)) = 3$  supporting Ulrich bundles  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})$  such that  $16a-12 \leq c_1(\mathcal{E})^2 \leq 18a$ . The above discussion shows that if  $c_1(\mathcal{E})^2 = 16a-12$ , then  $\mathcal{E} \cong \mathcal{O}_F(A) \oplus \mathcal{O}_F(B)$  with  $AB = 4a-2$ .

The next proposition is the first step in this direction.

**Proposition 5.1.** *Let  $a \geq 2$ . For each choice of an integer  $u$  in the range  $4a-2 \leq u \leq 5a+2$ , there exists a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  such that  $\text{Pic}(F)$  is freely generated by  $h, A, B$ , where  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are Ulrich line bundles.*

*Proof.* We fix the lattice  $\Lambda := \mathbb{Z}h \oplus \mathbb{Z}A \oplus \mathbb{Z}B$  having

$$M := \begin{pmatrix} 2a & 3a & 3a \\ 3a & 4(a-1) & u \\ 3a & u & 4(a-1) \end{pmatrix}$$

as its intersection matrix. Such a lattice is even. Moreover, it is an easy exercise to check that it has signature  $(1, 2)$  in the range  $4a-3 \leq u \leq 5a+3$ . Corollary 14.3.1 of [16] implies the existence of a projective  $K3$  surface  $F$  having  $\text{Pic}(F) \cong \Lambda$ .

Recall that for each divisor  $\Gamma$  with  $\Gamma^2 = -2$  on  $F$  we have the Picard–Lefschetz reflection  $\pi_\Gamma$  of  $\text{Pic}(F)$  defined by  $D \mapsto D + (D\Gamma)\Gamma$ . If  $D'$  is another divisor on  $F$  then  $\pi_\Gamma(D)\pi_\Gamma(D') = DD'$ , because  $\Gamma^2 = -2$ .

As pointed out in [16], Corollary 8.2.11, the cone of big and nef divisors is a fundamental domain for the group generated by the above reflections. In particular we can find divisors  $\Gamma_i$  with  $\Gamma_i\Gamma_j = -2\delta_{i,j}$ ,  $i = 1, \dots, \gamma$ , such that

$$h' := h + \sum_{i=1}^{\gamma} (h\Gamma_i)\Gamma_i$$

is nef. Let

$$A' := A + \sum_{i=1}^{\gamma} (A\Gamma_i)\Gamma_i, \quad B' := B + \sum_{i=1}^{\gamma} (B\Gamma_i)\Gamma_i.$$

Then  $h', A', B'$  generate  $\text{Pic}(F)$  and they still have  $M$  as intersection matrix. Omitting the prime in the superscript we can thus assume that  $h$  is nef.

We will now show that  $h$  is actually very ample. Since  $h^2 = 2a$  it will follow that the surface  $F$  can be embedded as a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$ .

Since  $h^2 = 2a \geq 4$ , thanks to Theorem 7 of [18] (see also [19], Lemma 2.4), we have to check that there are no effective divisors  $E$  on  $F$  satisfying anyone of the following conditions:

- $E^2 = 0$  and  $Eh = 1, 2$ ;
- $E^2 = 2$  and  $\mathcal{O}_F(h) \cong \mathcal{O}_F(2E)$ ;
- $E^2 = -2$  and  $Eh = 0$ .

Notice that, in any case, if  $E \in |xA + yB + zh|$ , then  $Eh = 3ax + 3ay + 2az$  is a multiple of  $a \geq 2$ .

Thus the first case can occur only if  $a = 2$  and, in this case,  $Eh = 2$  necessarily, i.e.  $2z = 1 - 3x - 3y$ . Simple computations show that

$$E^2 = -5x^2 - (18 - 2u)xy - 5y^2 + 1.$$

Consider the ellipse  $5x^2 + (18 - 2u)xy + 5y^2 = 1$ , where  $4a - 2 = 6 \leq u \leq 12 = 5a + 2$ . The  $x$ -coordinate intersection point of the ellipse with the line  $y = 1$  is a root of the polynomial  $5x^2 + (18 - 2u)x + 4$ .

The discriminant of this polynomial is  $\Delta(u) := u^2 - 18u + 61$  which is an increasing function for  $u \geq 9$  and symmetric around  $u = 9$ . Thus

$$u^2 - 18u + 61 \leq \Delta(12) = -11$$

in the whole range  $6 \leq u \leq 12$ .

We conclude that the line  $y = 1$  have no points in common with the ellipse. Since the ellipse is symmetric with respect to the origin, it immediately follows that it is strictly contained in the square with vertices  $(\pm 1, 0)$  and  $(0, \pm 1)$ : in particular there are no points with integral coordinates on the ellipse, because it does not pass through the origin.

The second case  $E^2 = 2$  cannot occur because  $h$  is an element of a basis of  $\text{Pic}(F)$  (see the comments after Lemma 2.4 of [19]).

Thus we look at the third case  $E^2 = -2$ . In this case equality  $Eh = 0$  implies  $2z = -3x - 3y$ . Simple computations show that

$$E^2 = -\frac{a+8}{2}x^2 - (9a-2u)xy - \frac{a+8}{2}y^2.$$

Consider the ellipse  $(a+8)x^2 + 2(9a-2u)xy + (a+8)y^2 = 4$ , where  $a \geq 2$  and  $4a - 2 \leq u \leq 5a + 2$ . Intersecting with the line  $y = 1$  we obtain  $(a+8)x^2 + 2(9a-2u)x + a+4 = 0$ . The discriminant is  $\Delta_a(u) := 4u^2 - 36au + 80a^2 - 12a - 32$  is an increasing function for  $u \geq 9a/2$  and it is symmetric around  $u = 9a/2$ . Thus

$$4u^2 - 36au + 80a^2 - 12a - 32 \leq \Delta_a(5a+2) = -4(a+4) < 0$$

in the range  $4a - 2 \leq u \leq 5a + 2$ .

We conclude that  $\mathcal{O}_F(h)$  is very ample, hence it embeds  $F$  in  $\mathbb{P}^{a+1}$  as a non-degenerate  $K3$  surface of degree  $2a$ . We will now prove that  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are Ulrich line bundles with respect to such an embedding. We restrict our attention to  $\mathcal{O}_F(A)$  because the argument for  $\mathcal{O}_F(B)$  is similar.

Since  $A^2 = 4(a-1) \geq 4$ , it follows from Equality (3) that either  $\mathcal{O}_F(A)$  or  $\mathcal{O}_F(-A)$  is effective. Since  $Ah = 3a \geq 6$  and  $h$  is very ample it follows that  $\mathcal{O}_F(A)$  is effective.

Assume that  $h^0(F, \mathcal{O}_F(A-h)) \neq 0$  and let  $D \in |A-h|$ . The divisor  $D$  is a curve of degree  $\deg(D) = (A-h)h = a$  such that  $D^2 = (A-h)^2 = -4$ , thus there is a proper integral subscheme  $E \subseteq D$  with  $E^2 = -2$ . Again let  $E \in |xA + yB + zh|$ . The degree of  $E$  is  $\deg(E) = (xA + yB + zh)h = 3ax + 3ay + 2az$ , hence it is a positive multiple of  $a$ .

It follows from the chain of inequalities  $a \leq \deg(E) \leq \deg(D) = a$  that  $\deg(E) = \deg(D)$ , whence  $E = D$ , a contradiction. We conclude that  $h^0(F, \mathcal{O}_F(A - h)) = 0$ .

Notice that  $3h - A$  enjoys the same intersections properties with  $h$  as  $A$ . Thus we can repeat the above discussion showing that  $h^0(F, \mathcal{O}_F(2h - A)) = 0$ . We conclude that  $\mathcal{O}_F(A)$  is Ulrich, by applying Lemma 4.1.  $\square$

**Remark 5.2.** It is not possible to extend the above proofs to the cases  $u = 4a - 3$  and  $u = 5a + 3$ . Indeed, in these cases,  $(A - B)h = (3h - A - B)h = 0$ . Moreover  $(A - B)^2 = -2$  in the first case, and  $(3h - A - B)^2 = -2$  in the second, thus  $\mathcal{O}_F(h)$  is not very ample. In these cases  $\mathcal{O}_F(h)$  embeds  $F$  as a singular surface.

We conclude the section with the following consequence of the above proposition. It shows that the bounds of Proposition 4.2 are actually sharp.

**Theorem 5.3.** *Let  $a \geq 2$ . For each choice of an integer  $u$  in the range  $4a - 1 \leq u \leq 5a + 4$ ,  $u \neq 5a + 3$ , there exists a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  and  $\text{rk}(\text{Pic}(F)) = 3$  supporting a simple Ulrich bundle  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  and  $c_2(\mathcal{E}) = u$ .*

*Proof.* Let  $u$  be an integer in the range  $4a - 1 \leq u \leq 5a + 2$  and  $F \subseteq \mathbb{P}^{a+1}$  a non-degenerate  $K3$  surface of degree  $2a$  containing divisors  $A$  and  $B$  with  $AB = u$  as in Proposition 5.1. Since  $u \neq 4(a - 1) = A^2 = B^2$ , it follows that  $\mathcal{O}_F(A) \not\cong \mathcal{O}_F(B)$ , thus the equalities  $(A - B)h = (B - A)h = 0$  and the ampleness of  $\mathcal{O}_F(h)$  imply

$$h^0(F, \mathcal{O}_F(A - B)) = 0, \quad h^2(F, \mathcal{O}_F(A - B)) = h^0(F, \mathcal{O}_F(B - A)) = 0.$$

Equality (3) for  $\mathcal{O}_F(A - B)$  implies

$$h^1(F, \mathcal{O}_F(A - B)) = -2 - \frac{(A - B)^2}{2} = AB - 4a + 2 \geq 1$$

We deduce the existence of the non-trivial Sequence

$$0 \longrightarrow \mathcal{O}_X(A) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_X(B) \longrightarrow 0,$$

thus  $\mathcal{E}$  is a rank 2 Ulrich bundle with  $c_1(\mathcal{E})^2 = (A + B)^2 = 8a - 8 + 2AB$  and  $c_2(\mathcal{E}) = AB$ .  $\mathcal{E}$  is simple, hence indecomposable, due to Lemma 2.3.  $\square$

**Remark 5.4.** The existence of special Ulrich bundles  $\mathcal{E}$  of rank 2 (i.e. such that  $c_1(\mathcal{E}) = 3h$ ) on each non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  follows from [1], Theorem 0.4, when  $\text{rk}(\text{Pic}(F)) = 1$ . For each bundle of this type  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  and  $c_2(\mathcal{E}) = u$  where  $u = 5a + 4$ .

Similarly, Theorem 4.6 of [19] implies the existence of non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  whose Picard group is freely generated by  $h$  and by a smooth irreducible curve  $A$  with  $Ah = 3a$  and  $A^2 = 4(a - 1)$ . The line bundle  $\mathcal{O}_F(A)$  is Ulrich (one can imitate the proof of the analogous statement in Proposition 5.1). As in the proof of Theorem 5.3 we still obtain special Ulrich bundles by extension of  $\mathcal{O}_F(A)$  by  $\mathcal{O}_F(3h - A)$ .

**Remark 5.5.** Notice that the non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  defined in Proposition 5.1 contains, besides the curves  $A$  and  $B$  with  $AB = u$ , also another curve, namely  $C \in |3h - A|$ : we have  $BC = 9a - u$ .

In particular for  $4a - 1 \leq u \leq 9a/2$  there is a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  supporting indecomposable Ulrich bundles  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 18a, 8a - 8 + 2u, 26a - 8 - 2u$ .

**Remark 5.6.** Each Ulrich bundle is semistable (see Theorem 2.2). Trivially the bundle  $\mathcal{E}$  constructed in Theorem 5.3 is strictly semistable, i.e. is semistable and not stable, because it contains a Ulrich line bundle  $\mathcal{O}_F(A)$  with  $\mu(\mathcal{O}_F(A)) = 3a = \mu(\mathcal{E})$ .

We showed that for each integer  $u$  in the range  $4a - 1 \leq u \leq 5a + 4$ ,  $u \neq 5a + 3$  there exists a  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  with  $\text{rk}(\text{Pic}(F)) = 3$  supporting Ulrich bundles  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  (see Theorem 5.3). It is obvious that such bundles cannot exist if  $\text{rk}(\text{Pic}(F)) = 1$ . We thus ask the following question.

**Question 5.7.** *If  $u$  is a fixed integer in the range  $4a - 1 \leq u \leq 5a + 4$ ,  $u \neq 5a + 3$ , is there a non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  with  $\text{rk}(\text{Pic}(F)) = 2$  supporting a Ulrich bundle  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$ ?*

As we already pointed out in Remark 5.4, the above question has an immediate positive answer when  $u = 5a + 4$ .

Moreover, for each integer  $u$  in the range  $4a - 1 \leq u \leq 9a/2$  there exist  $K3$  surfaces  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  supporting Ulrich bundles  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 18a, 8a - 8 + 2u, 26a - 8 - 2u$  (see Theorem 5.3 and Remark 5.5). It is quite natural to raise the following questions.

**Question 5.8.** *Does there exist a single non-degenerate  $K3$  surface  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  supporting Ulrich bundles  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  for each  $u$  in the range  $4a - 1 \leq u \leq 5a + 4$ ,  $u \neq 5a + 3$ ?*

**Question 5.9.** *If the answer to the previous question is positive, which is the minimal admissible value of  $\text{rk}(\text{Pic}(F))$ ?*

## 6. STABILITY OF GENERAL ULRICH BUNDLES OF RANK 2

If  $\mathcal{E}$  is a semistable bundle of rank 2 with reduced Hilbert polynomial  $p(t)$  (with respect to  $\mathcal{O}_F(h)$ ), then the coarse moduli space  $\mathcal{M}_F^{ss}(p)$  parameterizing  $S$ -equivalence classes of semistable rank 2 bundles on  $F$  with reduced Hilbert polynomial  $p(t)$  is non-empty (see Section 1.5 of [17] for the notion of  $S$ -equivalence). We will denote by  $\mathcal{M}_F^s(p)$  the open locus inside  $\mathcal{M}_F^{ss}(p)$  of stable bundles.

The scheme  $\mathcal{M}_F^{ss}(p)$  is the disjoint union of open and closed subsets  $\mathcal{M}_F^{ss}(2; c_1, c_2)$  whose points represent  $S$ -equivalence classes of semistable rank 2 bundles with fixed Chern classes  $c_1$  and  $c_2$ . Similarly  $\mathcal{M}_F^s(p)$  is the disjoint union of open and closed subsets  $\mathcal{M}_F^s(2; c_1, c_2)$ .

Grauert semicontinuity theorem for complex spaces (see [2]) guarantees that the property of being aCM in a family of vector bundles is an open condition.

Thus, on the one hand, we have open subschemes  $\mathcal{M}_F^{ss, aCM}(2; c_1, c_2) \subseteq \mathcal{M}_F^{ss}(2; c_1, c_2)$  and  $\mathcal{M}_F^{s, aCM}(2; c_1, c_2) \subseteq \mathcal{M}_F^s(2; c_1, c_2)$  parameterizing respectively  $S$ -equivalence classes of semistable and stable aCM bundles of rank 2 on  $F$  with Chern classes  $c_1$  and  $c_2$  (see Section 2 of [4]).

On the other hand, the locus of aCM bundles  $\mathcal{S}pl_F^{aCM}(r; c_1, c_2)$  inside  $\mathcal{S}pl_F(r; c_1, c_2)$  is open too. If we denote by  $\mathcal{S}pl_F^{ns, aCM}(2; c_1, c_2) \subseteq \mathcal{S}pl_F^{aCM}(2; c_1, c_2)$  the locus of simple aCM bundles which are not stable, then  $\mathcal{S}pl_F^{aCM}(2; c_1, c_2) \setminus \mathcal{S}pl_F^{ns, aCM}(2; c_1, c_2)$  is an open subset isomorphic to  $\mathcal{M}_F^{s, aCM}(2; c_1, c_2)$ .

We are interested in dealing with the moduli spaces of aCM bundles of rank 2 on  $F$ : in particular we are interested in those ones constructed in the previous section.

**Proposition 6.1.** *Let  $F \subseteq \mathbb{P}^{a+1}$  be a non-degenerate  $K3$  surface of degree  $2a$ , where  $a \geq 2$ .*

If  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are Ulrich line bundles on  $F$  such that  $4a - 1 \leq AB \leq 5a + 4$ ,  $AB \neq 5a + 3$ , then the moduli space  $\mathcal{M}_F^{ss,aCM}(2; A + B, AB)$  is non-empty.

If  $AB \geq 4a$ , for each irreducible component  $\mathcal{M} \subseteq \mathcal{M}_F^{ss,aCM}(2; A + B, AB)$ , the locus  $\mathcal{M} \cap \mathcal{M}_F^{s,aCM}(2; A + B, AB)$  is smooth and non-empty of dimension  $2AB - 8a + 2$ .

The points in  $\mathcal{M}_F^{ss,aCM}(2; A + B, AB) \setminus \mathcal{M}_F^{s,aCM}(2; A + B, AB)$  are in one-to-one correspondence with the unordered pairs  $\{ \mathcal{O}_F(\overline{A}), \mathcal{O}_F(\overline{B}) \}$  where  $\mathcal{O}_F(\overline{A})$  and  $\mathcal{O}_F(\overline{B})$  are Ulrich line bundles such that  $\mathcal{O}_F(\overline{A} + \overline{B}) \cong \mathcal{O}_F(A + B)$ .

*Proof.* Notice that Lemma 4.1 implies that the points in  $\mathcal{S}pl_F^{aCM}(2; A + B, AB)$  actually parameterize rank 2 simple Ulrich bundles with Chern classes  $A + B$  and  $AB$ .

We have that  $\mathcal{O}_F(A) \neq \mathcal{O}_F(B)$ , because  $AB > 4(a-1) = A^2$ . The locus  $\mathcal{S}pl_F^{ns,aCM}(2; A + B, AB)$  is non-empty, due to Theorem 5.3 and Remark 5.6, hence  $\mathcal{S}pl_F^{aCM}(2; A + B, AB)$  is smooth, non-empty and its dimension is  $4c_2 - c_1^2 - 6 = 2AB - 8a + 2$  (e.g. see [20], Theorem 0.1).

If  $\mathcal{E}$  is a bundle representing a strictly semistable Ulrich bundle in  $\mathcal{S}pl_F^{aCM}(2; A + B, AB)$ , then  $\mathcal{E}$  must contain a line bundle  $\mathcal{L}$  such that  $\mathcal{E}/\mathcal{L}$  is torsion free and  $\mu(\mathcal{L}) = \mu(\mathcal{E})$ . Theorem 2.2 implies that  $\mathcal{E}$  must fit into a sequence of the form

$$(6) \quad 0 \longrightarrow \mathcal{O}_F(\overline{A}) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_F(\overline{B}) \longrightarrow 0.$$

where  $\overline{A}$  and  $\overline{B}$  are Ulrich line bundles such that  $\overline{A} + \overline{B} = A + B$  and  $\overline{A}\overline{B} = AB$ . Thus we obtain a non-zero section of  $H^1(F, \mathcal{O}_F(\overline{A} - \overline{B})) \cong H^1(F, \mathcal{O}_F(2\overline{A} - A - B))$ .

As a first consequence of the above discussion, we see that  $\mathcal{S}pl_F^{ns,aCM}(2; A + B, AB)$  is dominated by the union of the projective spaces associated to  $H^1(F, \mathcal{O}_F(2\overline{A} - A - B))$  as  $\overline{A}$  varies in the subset of Ulrich line bundles inside  $\text{Pic}(F)$ . Such a set is trivially countable, thus  $\mathcal{S}pl_F^{ns,aCM}(2; A + B, AB)$  is a countable union of irreducible subschemes of dimension at most  $AB - 4a + 1$ : in particular, when  $AB \geq 4a$ , it cannot fill any irreducible component  $\mathcal{S} \subseteq \mathcal{S}pl_F^{aCM}(2; A + B, AB)$  (which is smooth of dimension  $4c_2 - c_1^2 - 6 = 2AB - 8a + 2$ ).

Since  $\mathcal{S}pl_F^{aCM}(2; A + B, AB)$  contains  $\mathcal{M}_F^{s,aCM}(2; A + B, AB)$  as an open subset, it follows that  $\mathcal{S} \setminus \mathcal{S}pl_F^{ns,aCM}(2; A + B, AB)$  is an irreducible component of  $\mathcal{M}_F^{s,aCM}(2; A + B, AB)$ . We deduce that the  $S$ -equivalent class of each  $\mathcal{E} \in \mathcal{S} \cap \mathcal{S}pl_F^{ns,aCM}(2; A + B, AB)$  is actually in the closure of a non-empty component of  $\mathcal{M}_F^{s,aCM}(2; A + B, AB)$ .

As a second consequence of the above discussion we obtain a characterization of the  $S$ -equivalence classes of each strictly semistable bundle  $\mathcal{E}$  in  $\mathcal{M}_F^{ss,U}(2; A + B, AB)$ . Indeed, if  $\mathcal{E}$  fits into Sequence (6), its Jordan-Hölder filtration is  $0 \subseteq \mathcal{O}_F(\overline{A}) \subseteq \mathcal{E}$ , thus the associated graded ring is

$$\text{gr}(\mathcal{E}) := \mathcal{O}_F(\overline{A}) \oplus \mathcal{E}/\mathcal{O}_F(\overline{A}) \cong \mathcal{O}_F(\overline{A}) \oplus \mathcal{O}_F(\overline{B}).$$

Let  $\mathcal{E}'$  be in the same  $S$ -equivalence class of  $\mathcal{E}$ . In particular  $\mathcal{E}'$  is strictly semistable, hence it fits into a sequence of the form

$$0 \longrightarrow \mathcal{O}_F(\overline{A}') \longrightarrow \mathcal{E}' \longrightarrow \mathcal{O}_F(\overline{B}') \longrightarrow 0,$$

where  $\mathcal{O}_F(\overline{A}')$ ,  $\mathcal{O}_F(\overline{B}')$  are Ulrich line bundles which are non-isomorphic. Thus

$$\mathcal{O}_F(\overline{A}') \oplus \mathcal{O}_F(\overline{B}') \cong \text{gr}(\mathcal{E}') \cong \text{gr}(\mathcal{E}) \cong \mathcal{O}_F(\overline{A}) \oplus \mathcal{O}_F(\overline{B}).$$

We have a non-zero morphism  $\mathcal{O}_F(\overline{A}) \rightarrow \mathcal{O}_F(\overline{A}') \oplus \mathcal{O}_F(\overline{B}')$ , thus either  $h^0(F, \mathcal{O}_F(\overline{A}' - \overline{A})) \neq 0$ , or  $h^0(F, \mathcal{O}_F(\overline{B}' - \overline{A})) \neq 0$ . In the first case, the equality  $(\overline{A}' - \overline{A})h = 0$  and the

ampleness of  $\mathcal{O}_F(h)$  imply  $\mathcal{O}_X(\overline{A} - \overline{A}) \cong \mathcal{O}_X$ , hence the above non-zero map induces an isomorphism  $\mathcal{O}_F(\overline{A}) \cong \mathcal{O}_F(\overline{A}')$ , thus

$$\mathcal{O}_F(\overline{B}') \cong \mathcal{O}_F(A + B - \overline{A}') \cong \mathcal{O}_F(A + B - \overline{A}) \cong \mathcal{O}_F(\overline{B}),$$

i.e.  $\{ \mathcal{O}_F(\overline{A}), \mathcal{O}_F(\overline{B}) \} = \{ \mathcal{O}_F(\overline{A}'), \mathcal{O}_F(\overline{B}') \}$ .

An analogous argument holds if  $h^0(F, \mathcal{O}_F(\overline{B}' - \overline{A})) \neq 0$ .  $\square$

The above proposition yields the following theorem (see also Proposition 3.22 of [10]).

**Theorem 6.2.** *Let  $F \subseteq \mathbb{P}^{a+1}$  be a non-degenerate  $K3$  surface of degree  $2a$ , where  $a \geq 2$ .*

*If  $\mathcal{E}$  is an indecomposable Ulrich bundle of rank 2 on  $F$  which is strictly semistable and whose  $S$ -equivalence class is a general point in its moduli space, then  $\mathcal{E}$  fits into Sequence (1) where  $\mathcal{O}_F(A)$  and  $\mathcal{O}_F(B)$  are Ulrich line bundles on  $F$  such that  $AB = 4a - 1$ . In particular  $c_1(\mathcal{E})^2 = 16a - 10$  and  $c_2(\mathcal{E}) = 4a - 1$ .*

*Proof.* If  $\mathcal{E}$  is strictly semistable, then it contains a line bundle  $\mathcal{O}_F(A)$  with  $\mu(\mathcal{O}_F(A)) = \mu(\mathcal{E}) = 3a$ . Trivially  $\mu(\mathcal{E}/\mathcal{O}_F(A)) = 3a$ : due to Theorem 2.2 it follows that  $\mathcal{O}_F(A)$  and  $\mathcal{E}/\mathcal{O}_F(A)$  are Ulrich line bundles. Then  $\mathcal{E}$  fits into Sequence (1).

If  $\mathcal{E}$  is also general in its moduli space, then Proposition 6.1 forces  $AB = 4a - 1$ .  $\square$

The construction of Section 5 yields the existence of semistable Ulrich bundles of rank 2 with  $c_1(\mathcal{E})^2 = 8a - 8 + 2u$  for each integer  $u$  in the range  $4a - 1 \leq u \leq 5a + 4$ ,  $u \neq 5a + 3$ .

On the one hand, Theorem 6.2 implies that when  $u \geq 4a$ , the general such bundle is actually stable. On the other hand, when  $u = 4a - 1$  the Proposition 6.1 implies that the moduli space is a finite set of points and we know that at least one of these points corresponds to a strictly semistable bundle.

**Question 6.3.** *Are there non-degenerate  $K3$  surfaces  $F \subseteq \mathbb{P}^{a+1}$  of degree  $2a$  supporting a stable Ulrich bundle  $\mathcal{E}$  of rank 2 with  $c_1(\mathcal{E})^2 = 16a - 10$ ?*

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